

Thermodynamics and Lemaitre-Tolman-Bondi void models

Priti Mishra* and Tejinder P. Singh†

*Department of Astronomy and Astrophysics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Colaba, Mumbai - 400005, Maharashtra, India*

(Dated:)

It has been argued in the literature that in order to make a matter dominated Friedmann-Lemaitre-Robertson-Walker universe compatible with the generalized second law of thermodynamics, one must invoke dark energy, or modified gravity. In the present article we investigate if in a similar spirit, inhomogeneous cosmological models can be motivated on thermodynamic grounds. We examine a particular minimal void Lemaitre-Tolman-Bondi inhomogeneous model which agrees well with observations. While on the one hand we find that the entropy associated with the apparent horizon is not well-behaved thermodynamically, on the other hand the canonical Weyl curvature entropy shows satisfactory thermodynamic behavior. We suggest that evolution of canonical Weyl curvature entropy might be a useful way to evaluate the thermodynamic viability of inhomogeneous cosmologies.

I. INTRODUCTION

Supernovae Ia observations in 1998 first hinted at the accelerated expansion rate of the universe [1, 2]. In a homogeneous and isotropic universe this accelerated expansion can be explained only by assuming a fluid with negative pressure. Such a fluid is referred to as dark energy. Ever since then there have been various candidates proposed for dark energy and also there have been various attempts to explain the accelerated expansion rate without invoking any dark energy component. The most popular and the simplest candidate for dark energy is cosmological constant (Λ). As the name suggests the energy density associated with cosmological constant does not evolve with time. One quantity which can play the role of cosmological constant is the vacuum because vacuum energy has the same equation of state as Λ . But the problem with Λ is that the value required to match with observation is smaller than its predicted value of vacuum energy from quantum field theory by a factor of 10^{120} . Therefore, the origin of cosmological constant is still unknown. And also the observed value of energy density of cosmological constant is of the same order as that of matter. This is called coincidence problem. There are other models of dark energy too namely, quintessence and k-essence models. In these models the dark energy evolves with space and time. Quintessence models solve the coincidence problem [3–5]. k-essence models have non-standard form of kinetic energy. In these models the universe is likely to go through big rip which means that the scale factor would become infinite some time in future and the large scale structures would be ripped apart and in some quintessence models there would come a time for freeze out which means the structures would collapse into each other.

Working within the framework of Einstein gravity dark energy is required only in homogeneous and isotropic uni-

verse. Since the scale of homogeneity is still not well established, it is perhaps interesting to relax this assumption of homogeneity of space and see whether dark energy is still required to explain the cosmological observations.

Many exact inhomogeneous solutions of Einstein equations have been found. Only Lemaitre-Tolman-Bondi (LTB), Szekeres and Stephani models have been studied in the context of cosmology. LTB is spherically symmetric dust solution of Einstein equations [6–8]. Szekeres is dust solution of Einstein equations [9]. Stephani metric is solution of Einstein equation when the energy density of the source has a pressure gradient [10]. There have been many inhomogeneous models proposed which attempt to explain the cosmological observations without assuming any dark energy component [11–31].

While the existence of dark energy was first hinted by SN Ia observations, there is a recent study which shows that for the generalized second law of thermodynamics to hold in Friedmann-Lemaitre-Robertson-Walker (FLRW) universe a dark energy component or something equivalent in modified gravity models must exist [32, 33]. The authors of [32, 33] defined the gravitational entropy as sum of matter entropy and the entropy of the apparent horizon. Apart from FLRW models, these authors investigated various other cosmological models and found that Chaplygin gas models [34, 35], some holographic dark energy models [36, 37], Dvali, Gabadadze and Porrati (DGP) brane model [38–40] and Cardassian model [41] are allowed by generalized second law of thermodynamics. Therefore the existence of dark energy is demanded not only by observations but also by thermodynamics.

Now it is interesting to see whether the inhomogeneous models which claim to explain the cosmological observations without dark energy satisfy the generalized second law of thermodynamics. For this we first need to define the gravitational entropy in inhomogeneous models. As a first step we follow the work of Pavon and Radicella by which we mean that we also define gravitational entropy as sum of the matter entropy and the entropy of the apparent horizon. But this definition may not be not universal because the apparent horizon may not exist in

* priti@tifr.res.in

† tpsingh@tifr.res.in

some inhomogeneous models while every gravitating system possibly has a gravitational entropy. Therefore, we explore other candidates of gravitational entropy as well. We consider Penrose's Weyl curvature hypothesis which proposes the relation between the gravitational entropy and Weyl tensor [42]. Following this hypothesis many different forms of entropy have been proposed [43–48]. And there have been various studies on the thermodynamics of various spacetimes [49–54]. We choose few candidates of entropy and also a realistic inhomogeneous model. By realistic inhomogeneous model we mean that such an inhomogeneous model which provide a good fit to the data. Then we investigate whether these candidates can represent gravitational entropy for the realistic inhomogeneous models.

The plan of this paper is as follows: Section II briefly explains Pavon and Radicella's work related to thermodynamics of cosmological models. In section III we briefly recall LTB models. Section IV describes the LTB void model for which we want to calculate the entropy. In section V we explain the apparent horizon and calculate the entropy of the LTB model. The results are shown in the same section. In section VI various other candidates of gravitational entropy are calculated in LTB model and the results are shown and interpreted. Then we discuss the results in section VII.

II. THERMODYNAMICS AND COSMOLOGICAL MODELS

The generalized second law of thermodynamics [GSLT] states that for a system to tend towards thermodynamic equilibrium its entropy should be ever increasing and its evolution should be of convex nature i.e. the entropy should satisfy the following two conditions: $S' > 0$ and $S'' < 0$ [55], where prime denotes an evolution parameter such as the scale factor or cosmological time. Pavon and Radicella [PR] have studied the thermodynamics of FLRW and some modified gravity models and argued that dark energy (or alternatively, modified gravity) is indicated on thermodynamic grounds [32]. Below we briefly describe PR's work:

PR proceed to calculate S' and S'' in FLRW models as follows: The entropy of a FLRW universe is sum of the entropy of the apparent horizon and the entropy of the fluids enclosed by the horizon. They chose the entropy of apparent horizon because the apparent horizon in FLRW universes always exists (which is not generally true for the particle horizon and the future event horizon) and is assumed to possess an entropy proportional to its area [56, 57] and also a temperature [58]. Besides, it appears to be the appropriate thermodynamic boundary [59]. Further, they assume that the entropy of the apparent horizon is proportional to its area via the following equation:

$$S_A \equiv \frac{k_B}{4} \frac{\mathcal{A}}{l_{Pl}^2} \quad (2.1)$$

where l_{Pl} and k_B are Planck length and Boltzmann constant respectively. \mathcal{A} is the surface area of the apparent horizon and can be calculated from

$$\mathcal{A} = 4\pi\tilde{r}_A^2 \quad (2.2)$$

where

$$\tilde{r}_A = (\sqrt{H^2 + (k/a^2)})^{-1}. \quad (2.3)$$

\tilde{r}_A is the radius of the apparent horizon and H the Hubble factor of the FLRW metric [56].

Using Friedmann equations and the conservation equation it can be shown that

$$\mathcal{A} = 4\pi\tilde{r}_A^2 = \frac{3}{2G} \frac{1}{\rho}, \quad (2.4)$$

and

$$\mathcal{A}' = \frac{9}{2G} \frac{1+w}{a\rho}, \quad (2.5)$$

where w is the equation of state parameter $p = w\rho$. Eqn. (2.5) shows that the area will increase in expanding universes if $1+w > 0$.

For $w = \text{constant}$ Eqn. (2.5) reduces to

$$\mathcal{A}'' = -\frac{9}{2G} \frac{1+w}{(a\rho)^2} (a\rho' + \rho) = \frac{9}{2Ga^2\rho} (1+w)(2+3w). \quad (2.6)$$

Above equation shows that $\mathcal{A}'' \leq 0$ for $-1 \leq w \leq -2/3$. Therefore, the FLRW models with equation of parameter in this range are not favored by the convexity condition of GSLT.

The entropy of the fluid enclosed by the apparent horizon is calculated from following equation:

$$S_m = k_B N \quad (2.7)$$

where N is the number of dust particles contained within the apparent horizon. This equation calls for an explanation. As noted by [32], since one is dealing with pressureless matter (i.e. dust), the matter is cold, and has an effective temperature $T = 0$, which does not permit the definition of an entropy in a traditional manner. Hence they propose associating one unit of entropy (k_B) with each particle. We can also motivate this somewhat differently, by considering the fundamental relation

$$TdS = dU + PdV \quad (2.8)$$

$$= k_B T dN + PdV \quad (2.9)$$

$$= k_B T dN + (\text{negligible term}) \quad (2.10)$$

at a constant temperature T , and by noting that because the matter is non-relativistic [dust], the pressure is extremely small, and the pressure term may be ignored. Then by comparing the left and right hand side, and treating the temperature as a very small but non-zero quantity which cancels out, Eqn. (2.7) follows, being the

same relation as proposed by [32]. In homogeneous and isotropic universe N should have the following form:

$$N = (4\pi/3)\tilde{r}_A^3 n \quad (2.11)$$

where $n = n_0 a^{-3}$ is the number density of dust particles. Substituting the form of N into the expression of matter entropy we obtain

$$S_m = k_B \frac{4\pi}{3} \tilde{r}_A^3 n_0 a^{-3} \quad (2.12)$$

which shows that the matter entropy is proportional to $a^{3/2}$. Hence S_m'' is positive. It shows that $S_m'' + S_A'' > 0$ in matter dominated FLRW universe, which is not in accordance with the GSLT.

Thus it has been shown by PR that in a pure matter FLRW universe the total entropy which is the sum of matter entropy and the entropy of the apparent horizon does not satisfy the convexity condition. When a dark energy component is added to the FLRW universe the entropy satisfies the generalized second law of thermodynamics. This dark energy can be the cosmological constant or a evolving dark energy i.e. quintessence or k-essence models.

PR also studied the thermodynamics of modified gravity models. They found that Chaplygin gas model [34, 35], some holographic dark energy models [36, 37], Dvali, Gabadadze and Porrati (DGP) brane model [38–40] and Cardassian model [41] also is in keeping with the generalized second law of thermodynamics.

III. LTB MODELS

The LTB metric is the spherically symmetric solution of Einstein equations for dust source. In comoving coordinates the LTB metric is given by:

$$ds^2 = c^2 dt^2 - \frac{\Phi'^2}{1-k} dr^2 - \Phi^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.1)$$

where $\Phi(t, r)$ is the area distance to the comoving shell at distance r from the origin and $k(r)$ is the curvature function.

Applying the LTB metric to the Einstein equations we obtain following equation:

$$\frac{1}{c^2} \dot{\Phi}^2 = \frac{2M}{\Phi} - k. \quad (3.2)$$

and

$$\kappa \rho c^2 = \frac{2M'}{\Phi^2 \Phi'}, \quad (3.3)$$

where $\kappa = 8\pi G/c^4$ and $M(r)$ is the gravitational mass within the comoving shell of radius r .

The Einstein Eqn. (3.2) has following solutions depending upon the sign of the curvature function $k(r)$:

Case 1. $k < 0$

$$\Phi = \frac{M}{(-k)} (\cosh \eta - 1), \quad (3.4)$$

$$c(t - t_B(r)) = \frac{M}{(-k)^{3/2}} (\sinh \eta - \eta) \quad (3.5)$$

Case 2. $k > 0$

$$\Phi = \frac{M}{k} (1 - \cos \eta), \quad (3.6)$$

$$c(t - t_B(r)) = \frac{M}{k^{3/2}} (\eta - \sin \eta) \quad (3.7)$$

Case 3. $k = 0$

$$\Phi = \left(\frac{9M}{2} \right)^{1/3} (t - t_B(r))^{2/3} \quad (3.8)$$

where $t_B(r)$ is called the bang time function. It tells the time of big bang for worldlines of comoving radius r . The case when k is positive, the evolution is called elliptic evolution because in this case the universe is closed which means that the the expansion of the universe would stop and after sometime it will start collapsing. The case when k is negative, the evolution is called parabolic. In this case universe is open and it will be ever expanding or ever collapsing. The last case when k is zero, the universe is flat which means that the expansion or the contraction will be everlasting. The LTB metric will yield FLRW metric if the mass density is homogeneous.

In LTB model there are three arbitrary functions out of which one is fixed by the coordinate choice since the equations so far are covariant under coordinate transformations.

IV. MINIMAL LTB VOID MODEL

We are interested in studying the evolution of the entropy in a realistic LTB void model which is a good fit to observations. There is one such minimal void model by Alexander et al. (AMV) which is good fit to SN Ia data and is also consistent with the WMAP 3-year data and the local measurements of Hubble parameter [23]. In this LTB void model the mass function, the curvature function and bang time function are defined as follows: given by

$$M(r) = \frac{1}{6} \bar{M}^2 M_P^2 r^3, \quad (4.1)$$

$$k(r) = -2(\bar{M}r)^2 k_{max} \left[1 - \left(\frac{r}{L} \right)^4 \right]^2, \quad (4.2)$$

$$t_B(r) = 0. \quad (4.3)$$

where M_P is the Planck mass, \bar{M} , k_{max} and L are parameters in the model. L is also the size of the LTB void beyond which the universe is described by FLRW metric. For the best fit, L is $250 \text{ Mpc} h^{-1}$ where h is .55. \bar{M} is determined from

$$\bar{M} = \frac{h_{out}}{3000} \sqrt{\frac{3}{8\pi}} \quad (4.4)$$

where h_{out} is the Hubble parameter in the FLRW region. k_{max} is a parameter related to the density contrast at the center of the LTB void.

V. THERMODYNAMICS OF AMV MODEL

Motivated by the work of PR we wish to see whether AMV model is in agreement with GSLT. For this we will calculate the gravitational entropy in this model and see whether the entropy satisfies both the conditions of GSLT. As a first step, following PR we also define entropy as sum of the entropy of the apparent horizon and the matter entropy. Before calculating the entropy of apparent horizon, we briefly recall the definition of apparent horizon and its mathematical expression for LTB models.

A. Apparent horizon

Since we want to calculate the entropy of apparent horizon, it is necessary to first define what an apparent horizon is and what is the mathematical condition for it to exist in a LTB universe. An apparent horizon is the outer envelope of a region of closed trapped surfaces. A closed trapped surface S_t is such a surface from which it is impossible to send a diverging bundle of light rays. At this surface both the outward-directed and the inward-directed bundles immediately converge. Mathematically, this statement can be written as: On S_t

$$k^\mu_{;\mu} \leq 0 \quad (5.1)$$

where k^μ is the tangent vector to the surface.

The mathematical condition for the apparent horizon in LTB spacetime is [60]:

$$\Phi = 2M. \quad (5.2)$$

After having written the mathematical expression of the apparent horizon in LTB models, we proceed to find to calculate the apparent horizon in AMV model using the following algorithm:

B. Algorithm to find the Apparent Horizon in AMV Model

Since AMV model contains two different spacetimes in the range $r < L$ and $r > L$, we will calculate the

apparent horizon in three steps: i) $r < L$, ii) $r = L$ and iii) $r > L$ where L is the size of the void.

Case 1: $r < L$ (Inside the void):

1. Since the curvature function $k(r)$ in AMV model is negative inside the void, the parametric solution of the Einstein equations in open LTB spacetime is given by

$$\Phi = \frac{M}{(-k)} (\cosh \eta - 1), \quad (5.3)$$

$$c(t - t_B) = \frac{M}{(-k)^{3/2}} (\sinh \eta - \eta) \quad (5.4)$$

where η is a parameter.

2. Since the area radius of the apparent horizon in LTB spacetime follows the condition given in eqn. (5.2), the above two equations reduce to the following two equations at $r = r_A$:

$$-2k = \cosh \eta - 1, \quad (5.5)$$

$$c(t - t_B) = \frac{M}{(-k)^{3/2}} (\sinh \eta - \eta). \quad (5.6)$$

3. We then eliminate η from eqns. (5.5) and (5.6) to find the following equation:

$$c(t - t_B) = \frac{M}{(-k)^{3/2}} (\sinh (\cosh^{-1} (1 - 2k)) - \cosh^{-1} (1 - 2k)) \quad (5.7)$$

Above equation can also be written as:

$$c(t - t_B) = \frac{M}{(-k)^{3/2}} \left[\left(\sqrt{(1 - 2k)^2 - 1} \right) - \cosh^{-1} (1 - 2k) \right] \quad (5.8)$$

4. In the above equation we substitute the functions $M(r)$, $k(r)$ and t_B for AMV model and obtain the time at which the apparent horizon lies at $r = r_A$ Mpc.

However, the above algorithm will work only when $r < L$ where L is the size of the LTB void because the sign of the curvature function $k(r)$ inside the void is different from that at the boundary of the void. Hence, the solution to the Einstein equations will also be different at the boundary.

Therefore, we use the following algorithm to calculate the apparent horizon at the boundary of the LTB void:

Case 2: $r = L$ (At the boundary of the void):

1. Since the curvature function is zero at the boundary, the area radius at the boundary is obtained from the following equation which is the solution to the Einstein equation for LTB spacetime with curvature function $k(r)$ as zero:

$$\Phi = \left(\frac{9M}{2}\right)^{1/3} (ct)^{2/3}. \quad (5.9)$$

2. Alternatively, one can also find the area distance by solving the Einstein equations for flat FLRW spacetime because $r = L$ is the interface between LTB and FLRW spacetimes in the AMV model. The solution to flat matter dominated FLRW tells that the area distance at the boundary is:

$$a(t)r = \left(\frac{3}{2}H_0\sqrt{\Omega_m t}\right)^{2/3} r. \quad (5.10)$$

3. We substitute Φ from Eqn. (5.9) into the Eqn. (5.2) and obtain the following equation:

$$t = \frac{4}{3c}M(r). \quad (5.11)$$

We calculate the apparent horizon at the boundary using above equation.

Case 3: $r > L$ (Outside the void):

1. In AMV model the universe is described by flat FLRW metric outside the LTB void. Therefore, we will use the solution of the matter dominated FLRW equation to find the apparent horizon. In matter dominated flat FLRW universe the scale factor is given by

$$a(t) = \left(\frac{3}{2}H_0\sqrt{\Omega_m t}\right)^{2/3}. \quad (5.12)$$

2. We substitute the functional form of the scale factor into the following equation:

$$a(t)r = 2M(r) \quad (5.13)$$

which is same as Eqn. (5.2) with Φ written in the form of its FLRW limit.

3. After substituting the scale factor into Eqn.(5.13), we obtain the following equation:

$$t = \left(\frac{2M(r)}{r}\right)^{3/2} \frac{2}{3H_0\sqrt{\Omega_m}}. \quad (5.14)$$

The above equation will give the radius of apparent horizon as a function of time in the FLRW region.

C. Entropy in Minimal LTB Void Model

After having found the area radius of apparent horizon, we calculate the gravitational entropy using the following algorithm:

1. Following PR we also calculate the entropy of the apparent horizon from the following equation:

$$S_A = \frac{k_B}{4} \frac{\mathcal{A}}{l_{Pl}^2} = \frac{k_B}{4} \frac{4\pi\Phi^2}{l_{Pl}^2}, \quad (5.15)$$

where \mathcal{A} is the surface area of the apparent horizon and l_{Pl} is Planck length.

2. Then we calculate the matter entropy as follows:
If there are N dust particles each of mass m , the matter entropy of the system would be given by

$$S_m = k_B N \quad (5.16)$$

3. Assuming that the dust particles are non-interacting and dividing the total mass contained within the apparent horizon by the mass of the dust particle we obtain the number of particles as

$$N = \frac{Mc^2/G}{m} \quad (5.17)$$

where m is the mass of the dust particle and $M(r)c^2/G$ is the total gravitational mass contained within the comoving shell of radius r .

4. Substituting N from above equation into the Eqn. (5.16) we obtain

$$S_m = k_B \frac{Mc^2/G}{m} \quad (5.18)$$

5. We assume $m = 10^{10} M_\odot$ because it is mass of a typical galaxy and galaxy is considered to play the role of dust particle at cosmological scales. However, we find that our results do not change when we vary m in a significant range around this value.

D. Results

Fig. 1 shows that the entropy of the apparent horizon is concave in nature. Fig. 2 displays the behavior of the matter entropy with time. Fig. 2 shows that the matter entropy increases almost linearly with time. This linear behavior was expected also. The matter entropy is proportional to the mass contained within the apparent horizon. According to Eqn. (5.2) this mass is proportional to the area radius of the apparent horizon which increases linearly with time. Hence, the double derivative of the matter entropy will be very small compared to that of the entropy of the apparent horizon (Fig.

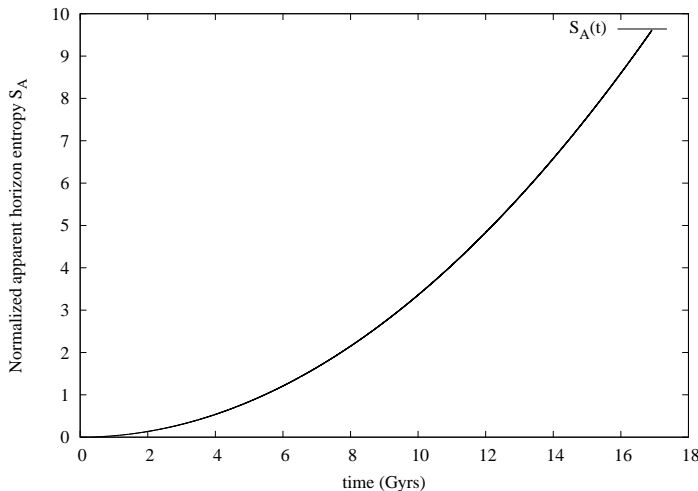


FIG. 1. Entropy of the apparent horizon vs. time where normalization factor is 10^{99} J/K.

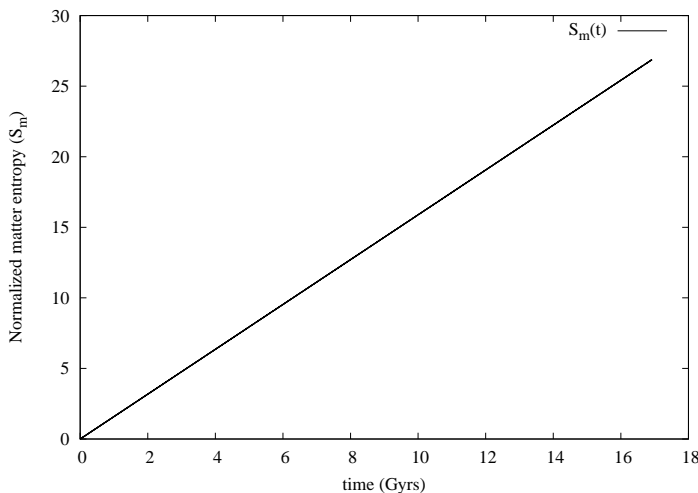


FIG. 2. Matter entropy vs. time where normalization factor is 10^{-12} J/K.

2). The total entropy is the sum of the matter entropy and the entropy of the apparent horizon. Since the apparent horizon entropy is of concave nature and double derivative of the matter entropy is very small, the sum of these two entropies will also be of concave nature.

Therefore, according to this definition of entropy this minimal void model does not satisfy the second condition of the generalized second law of thermodynamics. One could hence take the stance that the model is anti-thermodynamic or this definition is not a good representative of entropy in this minimal void model.

Another approach however would be to argue that inhomogeneous cosmological models which do not possess spatial symmetry may not have an apparent horizon at all. Keeping such a situation in mind, one could advocate looking for a more generic measure of gravitational en-

tropy, such as Weyl curvature. We investigate this next.

VI. WEYL TENSOR AND GRAVITATIONAL ENTROPY

The Weyl curvature hypothesis was proposed by Penrose. The starting point for the Weyl curvature hypothesis is the observation that the Big Bang singularity was very special. Penrose argues that the universe must have been in a low entropy state initially in order for there now to be a second law of thermodynamics. Assuming, as is commonly done, that the matter content of the universe was in thermal equilibrium near to the big bang, and therefore in a state of high entropy, one needs the contribution to the entropy from the rest of physics, which means the contribution from gravity or equivalently geometry, to be low. That is, the geometry must be highly ordered. Penrose introduced the concept of gravitational entropy and proposed that it is related to Weyl tensor. This hypothesis is called Weyl curvature hypothesis. The reason for proposing that the gravitational entropy is related to Weyl tensor because Weyl tensor is zero in a homogeneous state and it is non-zero for a universe with inhomogeneous structures. Therefore, if the gravitational entropy is related to Weyl tensor, it would evolve from a low entropy state to high entropy state which would be in accordance with the second law of thermodynamics.

Following the proposal of this hypothesis many candidates of the gravitational entropy have been proposed.

A. Candidates for gravitational entropy

We consider the following candidates for gravitational entropy motivated by the discussion in [53]:

1.

$$S = k_B l_{Pl} \int d^3x \sqrt{h} C \quad (6.1)$$

where C is Weyl scalar which is defined as

$$C = C^{abcd} C_{abcd}, \quad (6.2)$$

h is the determinant of the spatial part of the metric. Here and in the following, the fundamental length unit Planck length l_{Pl} has been introduced so as to obtain correct dimensions for entropy.

2. The standard canonical definition [44]

$$\delta s_c = k_B \frac{C_{abcd} C^{abcd}}{\mathcal{R}_{ab} \mathcal{R}^{ab}}, \quad (6.3)$$

where C_{abcd} is the Weyl tensor, and \mathcal{R}_{ab} is the Ricci tensor.

$$R_{ab} R^{ab} = \frac{4M'^2}{\Phi'^2 \Phi^4}. \quad (6.4)$$

and

$$C = C_{abcd}C^{abcd} = \frac{12}{\Phi^4} \left(\frac{2M'}{3\Phi'} - \frac{2M}{\Phi} \right)^2. \quad (6.5)$$

Substituting eqns. (6.4) and (6.5) into the Eqn. (6.3) we obtain

$$\delta s_c = k_B \frac{12\Phi'^2}{M'^2} \left(\frac{M'}{3\Phi'} - \frac{M}{\Phi} \right)^2. \quad (6.6)$$

3. The integrated version of the canonical definition:

$$\begin{aligned} S_c &= k_B / l_{Pl}^3 \int d^3x \sqrt{h} \delta s_c = \int d^3x \sqrt{h} \frac{C}{\mathcal{R}_{ab}\mathcal{R}^{ab}} \quad (6.7) \\ &= 4\pi k_B / l_{Pl}^3 \int_0^L dr \left\{ \frac{\Phi^2 \Phi'}{\sqrt{1-k}} \frac{12\Phi'^2}{M'^2} \left(\frac{M'}{3\Phi'} - \frac{M}{\Phi} \right)^2 \right\} \quad (6.8) \end{aligned}$$

4. The third candidate is the canonical definition multiplied by the square root of the determinant of the spatial metric [45],

$$S_h = k_B / l_{Pl}^2 \sqrt{h} \frac{C_{abcd}C^{abcd}}{\mathcal{R}_{ab}\mathcal{R}^{ab}} = \frac{\Phi^2 \Phi'}{\sqrt{1-k}} \frac{12\Phi'^2}{M'^2} \left(\frac{M'}{3\Phi'} - \frac{M}{\Phi} \right)^2. \quad (6.9)$$

We consider this candidate because it is shown in [45] that it is in accordance with the second law of thermodynamics. We investigate whether it can represent gravitational entropy in AMV model.

B. Results

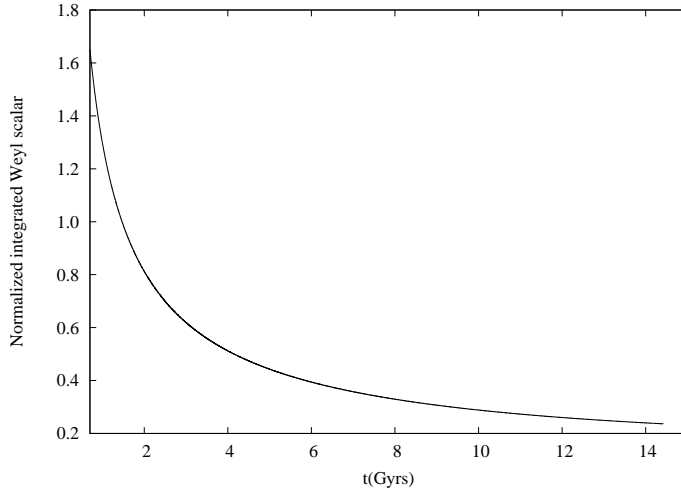


FIG. 3. Integrated Weyl scalar vs. time where normalization factor is 10^{-86} J/K.

Fig. 3 depicts the behavior of the first candidate of entropy Eqn. (6.1) with time. It shows that the entropy

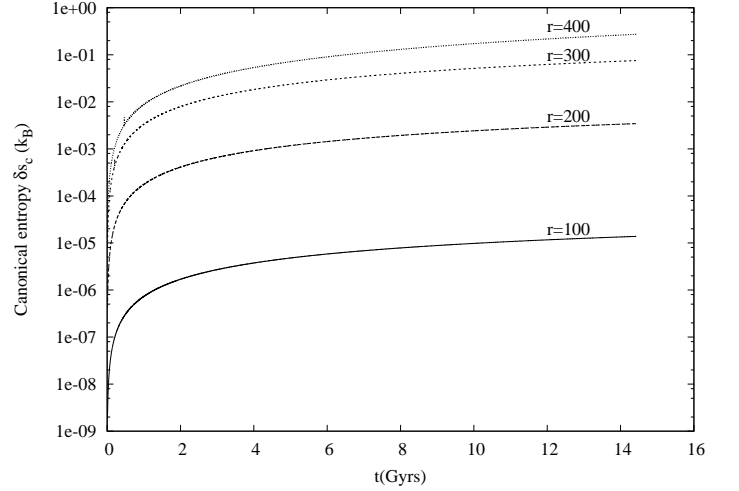


FIG. 4. Entropy (δs_c) vs. time where r is in Mpc.

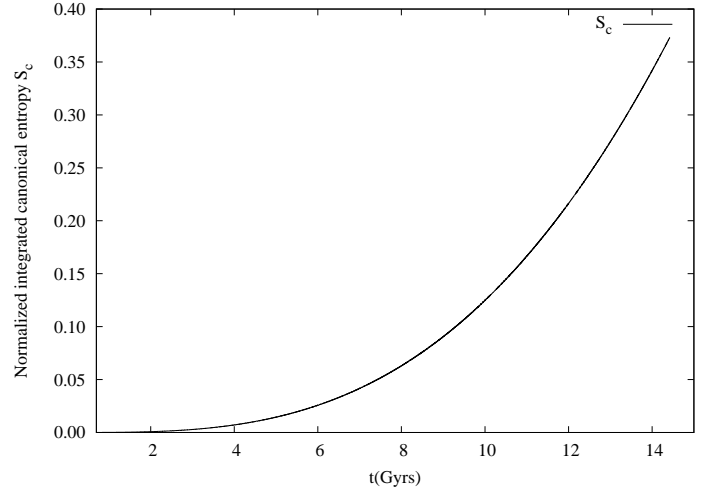


FIG. 5. Entropy S_c vs. time where normalization factor is 10^{157} J/K.

according to this definition is ever decreasing with time while we know that the entropy is ever increasing. Hence, the first candidate does not represent entropy for Alexander et al. LTB model. Figs. 4-6 display the time evolution of the other three candidates of entropy (eqns.(6.3), (6.8) and (6.9)). In Figs. 4 and 6 entropy has been plotted for many different values of r and all of them show same behavior.

We calculated four different candidates of the gravitational entropy proposed in the literature. We find that the first definition which is the integral of Weyl scalar over the volume of the void decreases with time. Hence, it does not represent the gravitational entropy. The second candidate which is the canonical definition of entropy increases linearly with time. When we add this entropy to matter entropy, the total entropy satisfies both condition of the second law of thermodynamics. Hence, the

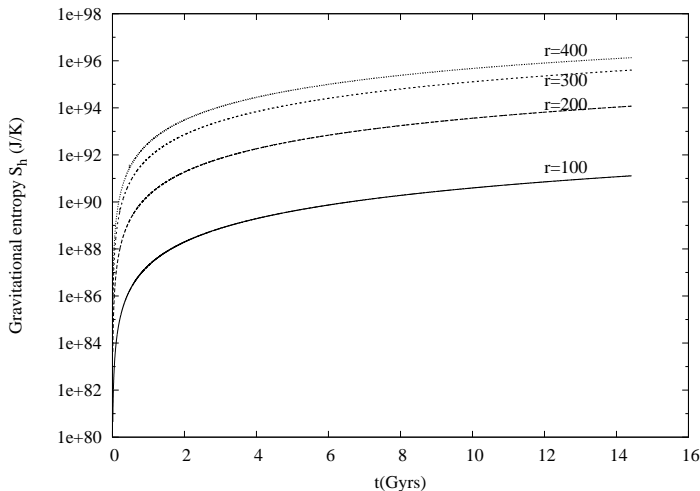


FIG. 6. Entropy S_h vs. time where r is in Mpc.

canonical definition represents the gravitational entropy in this LTB model. The third candidate is the integral of the first candidate over the volume of LTB void. This entropy increases with time but the entropy is of concave nature which does not satisfy the second condition of the GSLT which demands the convexity of entropy. Hence, the third candidate also does not represent the gravitational entropy. The fourth candidate also increases with time but it is not of convex nature. Hence, this candidate also does not represent the gravitational entropy.

Therefore, the canonical definition of Weyl entropy is a good representative of the gravitational entropy in the AMV model.

We would like to propose that further investigations of realistic inhomogeneous cosmological models should be carried out, to investigate the behavior of the canonical Weyl entropy. Such studies could help discriminate between different inhomogeneous models on thermodynamic grounds.

VII. DISCUSSION

We considered following five candidates for gravitational entropy:

1. The sum of the entropy of apparent horizon and matter entropy.
2. The integrated Weyl scalar over the LTB volume.
3. The canonical definition.
4. Integrated canonical entropy over LTB volume.
5. The product of canonical entropy by the square root of determinant of spatial metric.

We calculated these candidates of entropies for AMV model and found the following results.

1. The entropy of the apparent horizon is positive and increases with time. But its time evolution is concave in nature. The matter entropy increases almost linearly with time. Therefore, the sum of these two entropies would be of concave nature. Here, one should note that the matter entropy depends upon the mass of the dust particle. However, since the mass of the dust particle is just a multiplicative factor in the expression of matter entropy, its concave or convex nature will not change regardless of the mass of dust particle because the matter entropy increases linearly with time. Thus, we can infer that this candidate is not in keeping with second condition of GSLT. Therefore, it cannot be gravitational entropy for AMV model. However, we do not rule out this candidate for all inhomogeneous models. One should test this candidate for other models as well.
2. The integral of Weyl scalar over AMV volume is positive but decreases with time. It does not fulfill the first condition of GSLT which requires the entropy to be ever increasing. Hence, integrated Weyl scalar does not represent gravitational entropy in AMV model.
3. The canonical definition of gravitational entropy varies with comoving distance as well as with time. Therefore, we fix different values of r and study its evolution. We find that it shows same behavior for four different values of r in the large range 100-400 Mpc at interval of 100 Mpc. Hence, we can infer that the behavior of evolution of canonical entropy does not change with cosmological distance. Canonical entropy is positive and ever increasing. The rate of increase is almost linear and hence it fulfills the convexity condition of GSLT. Therefore, when this is added to the canonical entropy, the total entropy increase is in accordance with GSLT. Therefore, it can be a good representative of gravitational entropy in AMV model. However, this candidate should be tested for other inhomogeneous models as well to put stringent constraints on its being gravitational entropy candidate.
4. The integrated version of canonical definition of entropy is positive and is ever increasing also. However, the rate of increase is of concave nature. Hence, this candidate is not in keeping with the second condition of GSLT. Even when added to matter entropy its evolution will not be of convex because the matter entropy increases linearly with time and hence its second derivative with respect to time is zero and therefore it will not alter the overall behavior of total entropy. The nature of only the gravitational entropy candidate will decide whether the total entropy is concave or convex.
5. The last candidate which we studied is the product of canonical entropy and the square root of

the determinant of spatial metric. This candidate also depends upon time as well as on the comoving distance r . We fix four different values of r in the range 100-400 Mpc at interval of 100 Mpc and study its evolution. We find that for all these values this candidate shows same behavior. Hence, the evolution of this candidate does not depend on the cosmological distance. It is positive and increases monotonically. However, its evolution is of concave nature and hence it does not fulfill the convexity condition of GSLT. Therefore, this candidate also does not represent the gravitational entropy in AMV model.

We summarize our results by saying that out of all the five candidates of entropy only canonical definition fulfills all conditions of GSLT in AMV model. This candidate should further be studied for other inhomogeneous models.

Assuming that canonical definition is the correct definition of gravitational entropy AMV model is in keeping with the generalized second law of thermodynamics. Hence, dark energy is not required in a LTB void model by generalized second law of thermodynamics while it (or some alternative) is required in FLRW models (or mod-

ified gravity models).

Note added in proof. – It has been brought to our attention [61] that unlike what we stated at the start of Sec. IV above, the full analysis of the CMB spectrum in the AMV model was carried out in [26], and that the fitted local Hubble parameter is extremely low. Observational constraints on LTB models have been studied in [20], [27] and also in [62], [63]. While we agree that there are very strong observational constraints on LTB void models, one can nonetheless explore their thermodynamic properties, as has been done by us in this paper. It was pointed out to us [64] that gravitational entropy in LTB dust models has been studied also in [65] using the Clifton-Ellis-Tavakol and Hosoya-Buchert proposals for entropy.

ACKNOWLEDGMENTS

The tensor calculations were done using mathtensor package of Mathematica. We would like to thank James P. Zibin, Julien Larena, Roberto Sussman, and Thomas Buchert for their useful comments on our paper, and for bringing Refs. [62], [63], [65] and [66] to our attention.

-
- [1] Riess et al., *Astron. J.* **116**, 1009 (1998).
 - [2] Perlmutter et al., *Astrophys. J.* **565**, 586 (1999).
 - [3] Ivaylo Zlatev, Limin Wang, and Paul J. Steinhardt, *Phys. Rev. Lett.* **82**, 896 (1999).
 - [4] Takeshi Chiba, Antonio De Felice, and Shinji Tsujikawa, *Phys. Rev. D* **87**, 083505 (2013).
 - [5] Shruti Thakur, Akhilesh Nautiyal, Anjan A. Sen, T. R. Seshadri, *Mon. Not. R. Astron. Soc.* **427**, 988 (2012).
 - [6] G. Lemaitre, *Annales Soc. Sci. Brux. A* **53**, 51 (1933). [*Gen. Rel. Grav.* **29**, 641 (1997)].
 - [7] R. C. Tolman, *Proc. Nat. Acad. Sci.* **20**, 169 (1934) [*Gen. Rel. Grav.* **29**, 935 (1997)].
 - [8] H. Bondi, *Mon. Not. Roy. Astron. Soc.* **107**, 410 (1947).
 - [9] P. Szekeres, *Commun. Math. Phys.* **41**, 55 (1975).
 - [10] H. Stephani, *Class. Quantum Grav.* **4**, 125 (1987).
 - [11] Kenji Tomita, Hideki Asada, Takashi Hamana, *Prog. Theor. Phys. Suppl.* **133**, 155 (1999).
 - [12] M. N. C  lerier, *Astron. Astrophys.* **353**, 63 (2000).
 - [13] H. Iguchi, T. Nakamura, and K. Nakao, *Prog. Theor. Phys.* **108**, 809 (2002).
 - [14] H. Alnes, M. Amarsguoui and  . Gr  n, *Phys. Rev. D* **73**, 083519 (2006).
 - [15] D. J. H. Chung, and A. E. Romano, *Phys. Rev. D* **74**, 103507 (2006).
 - [16] K. Enqvist and T. Mattsson, *J. Cosmol. Astropart. Phys.* **02**, 019 (2007).
 - [17] K. Bolejko, *PMC Phys.* **A2**, 1 (2008).
 - [18] J. Garc  a-Bellido and T. Haugb  lle T, *J. Cosmol. Astropart. Phys.* **04**, 003 (2008).
 - [19] J. Garc  a-Bellido and T. Haugb  lle, *J. Cosmol. Astropart. Phys.* **09**, 016 (2008).
 - [20] J. P. Zibin, A. Moss and D. Scott, *Phys. Rev. Lett.* **101**, 251303 (2008).
 - [21] C.-M. Yoo, T. Kai, and K-i Nakao, *Prog. Theor. Phys.* **120**, 937 (2008).
 - [22] K. Enqvist, *Gen. Rel. Grav.* **40**, 451 (2008).
 - [23] S. Alexander, T. Biswas, A. Notari and D. Vaid, *J. Cosmol. Astropart. Phys.* **09**, 025 (2009).
 - [24] K. Bolejko and J. S. B. Wyithe, *J. Cosmol. Astropart. Phys.* **02**, 020 (2009).
 - [25] M. N. C  lerier, K. Bolejko and A. Krasinski, *Astron. Astrophys.* **518**, A21 (2010).
 - [26] T. Biswas, A. Notari and W. Valkenburg, *J. Cosmol. Astropart. Phys.* **11**, 030 (2010).
 - [27] A. Moss, J. P. Zibin and D. Scott, *Phys. Rev. D* **83**, 103515 (2011).
 - [28] S. Nadathur and S. Sarkar, *Phys. Rev. D* **83**, 063506 (2011).
 - [29] W. Godlowski, J. Stelmach and M. Szydlowski, *Class. Quantum Grav.* **21**, 3953 (2004).
 - [30] K. Bolejko and M.-N. C  lerier, *Phys. Rev. D* **82**, 103510 (2010).
 - [31] K. Bolejko and R. Sussman, *Phys. Lett. B* **697**, 265 (2011).
 - [32] Ninfa Radicella and Diego Pavon, *Gen. Rel. Grav.* **44**, 685 (2012).
 - [33] German Izquierdo and Diego Pavon, *Phys. Lett. B* **633**, 420 (2006).
 - [34] A. Y. Kamenschchik, U. Moschella, V. Pasquier, *Phys. Lett. B* **511**, 265 (2001).
 - [35] R. Jackiw, arXiv:physics/0010042
 - [36] W. Zimdahl, D. Pavon, *Class. Quantum Gravity* **24**, 5461 (2007).

- [37] I. Duran, D. Pavon, W. Zimdahl, *J. Cosmol. Astropart. Phys.* **07**, 018 (2010).
- [38] G. Dvali, G. Gabadadze, M. Porrati, *Phys. Lett.* **B485**, 208 (2000).
- [39] C. Deffayet, *Phys. Lett.* **B502**, 199 (2001).
- [40] C. Deffayet, G. Dvali, G. Gabadadze, *Phys. Rev.* **D65**, 044023 (2002).
- [41] K. Freese, M. Lewis, *Phys. Lett.* **B540**, 1 (2002).
- [42] R. Penrose, *General Relativity: An Einstein Centenary Survey*, Cambridge University Press (1979) p. 581.
- [43] S. W. Goode and J. Wainwright, *Class. Quantum Grav.* **2**, 99 (1985).
- [44] J. Wainwright and P. J. Anderson, *Gen. Rel. Grav.* **16**, 609 (1984).
- [45] ØGrøn and S. Hervik, *Class. Quant. Grav.* **18**, 601 (2001).
- [46] A. Hosoya, T. Buchert, M. Morita, *Phys. Rev. Lett.* **92**, 141302 (2004).
- [47] R. Sussman, *Class. Quant. Grav.* **30**, 065015 (2013).
- [48] T. Clifton, G. F. R. Ellis, R. Tavakol, *Class. Q. Grav.* **30**, 125009 (2013).
- [49] S. Barve and T. P. Singh, *Mod. Phys. Lett.* **A12**, 2415 (1997).
- [50] S. Barve and T. P. Singh, arXiv:9705060.
- [51] M. D. Pollock and T. P. Singh, *Class. Quantum Grav.* **6**, 901 (1989).
- [52] Rong-Gen Cai, Li-Ming Cao and Ya-Peng Hu, *Class. Quantum Grav.* **26**, 155018 (2009).
- [53] Krzysztof Bolejko and William R. Stoeger, *Phys. Rev.* **D88**, 063529 (2013).
- [54] Nairwita Mazumder, Ritabrata Biswas and Subenoy Chakraborty, *Gen. Rel. and Grav.* **43**, 1827 (2011).
- [55] H.B. Callen, *Thermodynamics* (J. Wiley, NY, 1960).
- [56] D. Bak, S.-J. Rey, *Class. Quantum Grav.* **17**, L83 (2000).
- [57] R.-G. Cai, *Progr. Theoret. Phys. Suppl.* **172**, 100 (2008).
- [58] R.-G. Cai, L.-M. Cao, and Y.-P. Hu, *Class. Quantum Grav.* **26**, 155018 (2009).
- [59] B. Wang, Y. Gong, and E. Abdalla, *Phys. Rev.* **D74**, 083520 (2006).
- [60] J. Plebanski and A. Krasinski, *An Introduction to General Relativity and Cosmology*, Cambridge University Press (2006).
- [61] J. P. Zibin (private Communication).
- [62] V. Marra and M. Paakkonen, *J. Cosmol. Astropart. Phys.* **12** (2010) 021.
- [63] P. Bull, T. Clifton, and P. G. Ferreira, *Phys. Rev. D* **85**, 024002 (2012).
- [64] J. Larena (private Communication).
- [65] R. A. Sussman and J. Larena, *Classical Quantum Gravity* **31**, 075021 (2014).
- [66] N. Li, T. Buchert, M. Morita, and D. J. Schwarz, *Phys. Rev. D* **86**, 083539 (2012).